Project report on some basic topics of mathematics

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August 14, 2014

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Abstract

This is an project report about some basic concepts in mathematics including curve tracing, vector spaces, analysis which I studied under Dr. Shameek Paul of Centre for Excellence in Basic Sciences(UM-DAE CBS), Mumbai as a guide during the period of time from 20 June 2014 to 20 July 2014. I would like to thank Dr. Shameek Paul by giving his valuable time to guide me.

Signature of Guide
Dr. Shameek Paul

Signature of Student
Jishu Das

Chapter 1

Curve tracing

1.1 Some Basics

For $f : I \to R$ if $f'(c) > 0$and $f\prime$ is continuous at c for some $c \in I$, then f is increasing in some open interval containing c (not necessarily in I). If $f\prime(c) < 0$, then f is decreasing in some open interval containing c. Not necessarily in I can be clear by the given example below.

Let $f : [0, 1] \to R$

$f(x) = x^2 - \frac{x}{2}$, $f'(x) = 2x - \frac{1}{2}$,$c = \frac{1}{2}$ and $f\prime(\frac{1}{2}) = \frac{1}{2}$

For $f'(x) > 0$ we get an interval $(\frac{1}{4}, 1)$ which is not equal to I

It is important to note that we are writing $f'(x) > 0$ i.e. function is increasing in $(\frac{1}{4}, 1)$ not in $(\frac{1}{4}, 1]$. this is because our function is defined over $[0, 1]$. we do not know the behaviour of the function for some $x > 1$ it may so happen that $f(1) < f(x)$ for some open interval $(1, 1 + \epsilon)$ and $\epsilon > 0$ which will suggest f is not increasing at $x = 1$.

Definition 1:

Let $f : (a, b) \to R$ be differentiable and $c \in (a, b)$. Let $l(x) = f'(c)(x-c)+f(c)$be the tangent line to$f$ at point $(c, f(c))$

Definition 2:

The graph of f is said to be convex or concave at $(c, f(c))$ if there exists an open interval J containing c such that $\forall x \in$ J and $x \neq$ c $f(x) > l(x)$ or $f(x) < l(x)$ respectively.

Definition 3:

The point $(c, f(c))$ is said to be a point of inflection of f if there exists an open interval I containing c such that either

$f''(x) < 0$ if $x < c$ and $f''(x) > 0$ if $x > c$
or
$f''(x) > 0$ if $x < c$ and $f''(x) < 0$ if $x > c$

1.2 Some Basic results from Analysis

Result 1:

Intermediate Value Theorem :- Let $f : [a, b] \to R$ be continuous and let $d$be a real number in between $f(a)$ and $f(b)$, then there exists a point c such that $f(c) = d$.

Result 2:

Mean Value Theorem :- Let $f : [a, b] \to R$ be differentiable on (a, b) and continuous at a and b, then there exists a point c in between a and b such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Result 3:

A function $f : X \to R$ is said to be continuous at x if and only if for every sequence $x_n : X \cap N \to R$ and $lim\ x_n \to x \Rightarrow lim\ f(x_n) \to f(x)$

If there exists a sequence such that $x_n \to x \Rightarrow f(x_n) \to f(x)$ is not true then f is discontinuous at x.

If function $f : X \to R$ is differentiable at some $x \in X$ then it is continuous at x. But not necessarily the converse.

Proof :- f is differentiable at x

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x)(exists)$$

$$\Rightarrow \lim_{h \to 0} \frac{f(x+h) - f(x) - hf'(x)}{h} = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{g(h)}{h} = 0$$

$g(h) = f(x+h) - f(x) - hf'(x)$

In order to have the limit exist $g(h) = 0$

Now as $x_n \to x \Rightarrow x_n - x \to 0$

By taking $h = x_n - x$ we have

$$\lim_{n\to\infty} g(x_n - x) = \lim_{x_n \to x} g(x_n - x) = 0$$
$$\Rightarrow \lim_{x_n\to x} f(x_n) - f(x) - f'(x)(x_n - x) = 0$$
$$\Rightarrow \lim_{x_n\to x} f(x_n) - f(x) = 0$$

as $x_n - x \to 0$ and $f'(x)$ exists

$$\forall x_n \to x \Rightarrow f(x_n) \to f(x)$$

This shows that function is continuous at x. Note that the converse is not true i.e. if a function is continuous then it is necessary that it will be differentiable. we can take $f : R \to R$ and $f(x) = |x|$

Result 4:

Let $f : [a, b] \to R$ be continuous over $[a, b]$ and $c \in (a, b)$ and $f(c) > 0$, There exists an open interval I containing c such that $\forall x \in I \Rightarrow f(x) > 0$

Proof :- Let us construct a sequence such that there exists an x_n such that $f(x_n) \leq 0$. Consider an open Interval $(c - 1, c + 1)$ containing c such that it contains atleast one point x_1 such that $f(x_1) \leq 0$. Now decrease the open interval to $(c-\frac{1}{2}, c+\frac{1}{2})$ and there exists atleast one point x_2 such that $f(x_2) \leq 0$. In similar manner we can take upto n intervals.

For x_1 we have $|x_1 - c| < 1$ similarly for x_2 we have $|x_2 - c| < \frac{1}{2}$ likewise
$0 < |x_n - c| < \frac{1}{n}$ we know $\frac{1}{n} \to 0$ as $n \to \infty$
$\Rightarrow x_n \to c$ as $n \to \infty$ by sandwich theorem.
As f is continuous at c for $x_n \to c \Rightarrow f(x_n) \to f(x)$
$f(c) > 0$ implies that there exists an open interval I' such that for $n > n_0$ $f(x_n) > 0$. This contradicts that there exists an interval for which at least at one point in that interval $f(x) \leq 0$.

Result 5:

Let $f : [a, b] \to R$ be differentiable on (a, b), continuous at a and b , let $g = f'$, $c \in (a, b)$ and g is continuous on $[a, b]$ such that $f(c) = 0$ and $f'(c) > 0$,$f(x)$ is not a constant function and for some point $x_1 < c$ and $c < x_2$ inside some open interval I containing c we have $f(x_1) < (f(c) = 0)$ and $f(x_2) > (f(c) = 0)$.

Proof :- $g = f'$ and g is continuous on $[a, b]$ then we can say there exists an open interval I containing c such that $\forall x \in I \Rightarrow f(x) > 0$. Choose an interval I_1 between I as $[x_1, c]$. On applying mean value theorem on $[x_1, c]$ we have there exists a point a_1 inside I_1 such that $\frac{f(c)-f(x_1)}{(c-x_1)} = f'(a_1)$
$f'(a_1) > 0$ as I_1 is in between I and $c - x_1 > 0 \Rightarrow f(c) - f(x_1) > 0$

$\Rightarrow f(x_1) < 0$ as $f(c) = 0$.

similarly by taking an interval I_2 between I as $[c, x_2]$ and applying mean value theorem to it we can show that $f(x_2) > 0$. Now on combining we can write $f(x_1) < f(c) < f(x_2)$ for some x_1 and x_2 inside I_1 which contradicts $f(x_1) = f(c) = f(x_2)$ i.e. f is a constant function.

1.3 Theorem

Theorem 1:
Let $f : (a, b) \to R$ be differentiable and f'' is continuous. Assume that $f''(c)$ exists at $c \in (a, b)$. Then
(1) If $f''(c) > 0$, then f is convex at c.
(2) If $f''(c) < 0$, then f is concave at c.
(3) If c is a point inflection, then $f''(c) = 0$. Not necessarily the converse is true.

Proof :- In order to show f is convex at c, we need to show that $f(x) > l(x) \forall x \neq c$ in an open interval containing c.

$$f(x) - l(x) = f(x) - f(c) - f'(c)(x - c) \tag{1.1}$$

We apply mean value theorem to f either on $[c, x]$ if $x > c$ or on $[x, c]$ if $x < c$. In each case there exists a point x_1 between x and c such that

$$f(x) - f(c) = f'(x_1)(x - c) \tag{1.2}$$

Substituting equation (1.2) in equation (1.1) we get

$f(x) - l(x) = [f'(x_1) - f'(c)](x - c)$

Now let $g = f'$ and $h = g'$ apply Result 4 for h interval (a, b). h is continuous on this interval, $h(c) > 0$ and $c \in (a, b) \Rightarrow$ There exists an open interval I containing c such that $\forall y \in I f(y) > 0$.

Now apply Result 5 by taking $g'(c) > 0$ for the open interval I. Lets take $x < c$ we have $g(c) > g(x_1)$ for $c > x_1 \Rightarrow (f'(x_1) - f'(c)) < 0 \text{and} x - c < 0$

$\Rightarrow (f'(x_1) - f(c))(x - c) > 0$

$\Rightarrow f(x) - l(x) > 0$ or $f(x) > l(x)$

for $x > c$ we have $g(c) < g(x_1)$ for $c < x_1$ and in similar manner we can prove that $f(x) > l(x)$.

(2) is proved similarly. We will prove (3) shortly.

Point of Inflection

(1) Where f'' changes sign
(2)Geometrically the graph changes from convexity to concavity or vice versa at that point.
(3)If $x = f(t)$ where x is the position of a moving particle then point of inflection is the time at which the particle switches from acceleration to deceleration or vice versa.
(4)At point of inflection tangent line at c crosses the graph of function. (loosely not equivalent to the definition of inflection point.

Let us try to proof Theorem part (3) by fourth point of point of inflection. c is a point of inflection then the graph of the function crosses the tangent at c. If f is concave at c then graph of f lies below the tangent line at c in an open interval around c. So f can not be concave at c. $f''(c)$ not less than 0 (From theorem part-1). Similarly by taking convexity we can prove $f''(c)$ is not greater than 0 . This implies $f''(c) = 0$

1.4 Asymptote

Definition 4:

An affine (called "linear" in analysis) function $l(x) = ax + b$ is called asymptote for f as $x \to \infty$ if

$$\lim_{x\to\infty} (f(x) - l(x)) = 0$$

A vertical line $x = a$ is an asymptote for f if $lim_{x\to a_+} f(x)$ is infinite or $lim_{x\to a_-} f(x)$ is infinite.

For $f(x) = \frac{p(x)}{q(x)} = \frac{(a_n x^n + \ldots.. + a_0)}{(b_m x^m + \ldots.. + b_0)}$ be a rational function with a_n and b_m non zero.

(1) If degree of $p(x) = n < m =$ degree of $q(x)$, then f has the $x-$axis as an asymptote.
(2)If $n = m$ then f has a horizontal asymptote $l(x) = \frac{a_n}{b_m}$.
(3)If $n = m + 1$, we divide p and q and write $f(x) = ax + b + c(x)$ where $\lim_{x\to\infty} c(x) = 0 = \lim_{x\to-\infty}$. Then the line $l(x) = ax + b$ is a linear asymptote for f.
(4)If $n > m + 1$, then f has no non-vertical linear asymptote.
(5)The vertical asymptote for a rational function corresponds to the zeros of the denominator.

1.5 Tips for sketching the curve

(1) Symmetry
(a)Odd function if $\forall x \in domf, f(-x) = -f(x)$. Graph of f is symmetric about $y = -x$.
(b)Even function if $\forall x \in domf, f(-x) = f(x)$. Graph of f is symmetric about

$x = 0$ i.e. $y-$axis.
(c)Periodic function if $\forall x \in dom f$ and for a particular fundamental a. The look of graph remains the same as it appears inside the fundamental period.

(2)Placing the points
(a) Look for the point $(0, f(0))$.
(b)See the behaviour of f when $x \to -\infty$ and $x \to \infty$
This is just like we are watching how the function behaves when $x \to -\infty, 0, \infty$.Just like when we have to sketch some function under some closed interval $[a, b]$ we check the functional value at $x = a, b, \frac{a+b}{2}$.

(3)Values of x for which f is not defined. Study the behaviour of the function near those values of x.

(4)Locating the critical points
(a)Find out at which point f is having local maxima and minima. Obtain the interval for which f is decreasing or increasing.
(b)Find out point of inflection and interval for which f is concave or convex.

Chapter 2

Vector Spaces

2.1 Some basics

Definition 1:

A non-empty set V is said to be vector space over R (or a real vector space) if there exist maps $+ : V \times V \to V$, defined by $(x, y) \mapsto x+y$, called *addition* and $\cdot : R \times V \to V$, defined by $(\alpha, x) \mapsto \alpha \cdot x$, called *scalar multiplication*, satisfying the following properties:

(i)$x + y = y + x$ (commutativity of addition).
(ii)$(x + y) + z = x + (y + z)$ (associativity of addition).
(iii)There exists $\mathbf{0} \in V$ such that $x + \mathbf{0} = x = \mathbf{0} + x$ (existence of additive identity).
(iv)For every $x \in V$ there exists $y \in V$ such that $x + y = \mathbf{0} = y + x$. This y is denoted by $-x$ (existence of additive inverse).
(v)$\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$.
(vi)$(\alpha + \beta) \cdot x = \alpha \cdot x + \alpha \cdot y$.
(vii)$(\alpha\beta) \cdot x = \alpha(\beta \cdot x)$.
(viii)$1 \cdot x = x$.
We will adopt the following standard notation: $x + (-y)$ is written as $x - y$ $\forall x, y \in V$ and for $\alpha \in R$ and $x \in V$ we write αx for $\alpha \cdot x$.

Theorem 1: In a vector space V, we have
(a) $0 \cdot x = \mathbf{0}$ $\forall x \in V$.

Proof :- $0 \cdot x = (0 + 0) \cdot x = 0 \cdot +0 \cdot x$ equation(1)(by (iii) and (vi))
Adding $-0 \cdot x$ (by (iv)) on both sides we get
$\mathbf{0} = 0 \cdot x + (-0 \cdot x)$ ((by (iv))
$= (0 \cdot x + 0 \cdot x) + (-0 \cdot x)$ (by equation(1))
$= 0 \cdot x + (0 \cdot x + (-0 \cdot x))$ (by (ii))
$= 0 \cdot x + \mathbf{0}$ (by(iv)) $= 0 \cdot x$ (by (iii))

(b) There is unique additive identity. It means, if $\mathbf{0}$ and $\mathbf{0}'$ are such that $x+\mathbf{0} = x$ and $x + \mathbf{0}'$ $\forall x \in V$, then $\mathbf{0} = \mathbf{0}'$

Proof :-$x + \mathbf{0} = x = x + \mathbf{0}'$ $\forall x \in V$. In particular we can write $\mathbf{0} + \mathbf{0}' = \mathbf{0}$ as $\mathbf{0}'$ is an additive identity.Also $\mathbf{0} + \mathbf{0}' = \mathbf{0}'$ as $\mathbf{0}$ is an additive identity.
$\Rightarrow +\mathbf{0}' = \mathbf{0} + \mathbf{0}' = \mathbf{0}$
$\Rightarrow +\mathbf{0}' = +\mathbf{0}$

(c)The additive inverse is unique. If for a given x there are $y, y' \in V$ such that $x + y = 0$ and $x + y' = 0$, then $y = y'$.

Proof :- $x + y = \mathbf{0} = x + y'$. Adding y' to the first two sides we have
$y' + (x + y) = y' + \mathbf{0}$
$\Rightarrow (y' + x) + y = y' + \mathbf{0}$
$\Rightarrow \mathbf{0} + y = y'$
$\Rightarrow y = y'$.

(d)$(-1 \cdot x) = -x$, the negative element such that $x + (-x) = \mathbf{0}$ $\forall x \in V$

Proof :- $(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = (-1 + 1) \cdot x = 0 \cdot x = \mathbf{0}$ so that $(-1) \cdot x = -x$.

(e) $\alpha \cdot \mathbf{0} = \mathbf{0}$ $\forall \alpha \in R$ and $\mathbf{0} \in V$.

Proof :- We have $\mathbf{0} = \mathbf{0} + \mathbf{0}$
$\Rightarrow \alpha \cdot \mathbf{0} = \alpha \cdot (\mathbf{0} + \mathbf{0}) = \alpha \cdot \mathbf{0} + \alpha \cdot \mathbf{0}$
$\Rightarrow \alpha \cdot \mathbf{0} = \alpha \cdot \mathbf{0} + \alpha \cdot \mathbf{0}$
Adding $-\alpha \cdot \mathbf{0}$ to both sides
$\Rightarrow \alpha \cdot \mathbf{0} + (-\alpha \cdot \mathbf{0}) = (\alpha \cdot \mathbf{0} + \alpha \cdot \mathbf{0}) + (-\alpha \cdot \mathbf{0})$
$\Rightarrow \mathbf{0} = \alpha \cdot \mathbf{0} + (\alpha \cdot \mathbf{0} + (-\alpha \cdot \mathbf{0}))$
$\Rightarrow \mathbf{0} = \alpha \cdot \mathbf{0} + \mathbf{0}$
$\Rightarrow \mathbf{0} = \alpha \cdot \mathbf{0}$.

(f)If $\alpha \cdot x = 0$ $\forall \alpha \in R$ and $x \in V$, then either $\alpha = 0$ or $x = 0$.

Proof :- If $\alpha \cdot x = \mathbf{0}$ and $\alpha \neq \mathbf{0}$, then we can multiply both sides of $\alpha \cdot x = \mathbf{0}$ by α^{-1} to get
$\alpha^{-1} \cdot (\alpha \cdot x) = \alpha^{-1} \cdot \mathbf{0}$
$\Rightarrow (\alpha^{-1} \cdot \alpha) \cdot x = \alpha^{-1} \cdot \mathbf{0}$
$\Rightarrow x = \mathbf{0}$

Example :- Let X be a non-empty set. Let $V = F(X, R) = f : X \to R$ be the set of all real valued functions on the set X. In order to show that V is a vector space we need to check that it should satisfy all the property. So for two real valued function $f, g \in V$, we need to define $f + g$.
We define it as follows $(f + g)(x) := f(x) + g(x)$.$f : X \to R$ and $g : X \to R$ $\Rightarrow (f + g)(x)$ is defined for the common domain of $f(x)$ and $g(x)$ i.e. on set X. $f(x) \in R$ and $g(x) \in R$ $\forall x \in X \Rightarrow f(x) + g(x) \in R$ $\forall x \in X$. So $f + g : X \to R$ real valued function.That is $f + g \in V$.
We define $(\alpha f)(x) := \alpha f(x)$ $\forall \alpha \in R$. Domain of $(\alpha f)(x)$ is same as that of $f(x)$. $\alpha f(x) \in R$ $\forall \alpha \in R$ and $x \in X$. So $(\alpha f) : X \to R \Rightarrow \alpha \in V$.
The $\mathbf{0}$ for the vector space V can be defined as $\theta : X \to R$, $\theta(x) = 0$.

Similarly we can prove those eight properties.

2.2 Subspaces

Definition 1:

Let W be a non empty subset of vector space V, then W is said to be a subspace of V if W itself is a vector space under the operation induced from V.

Definition 2:

Let S be a non empty subset of vector space V. We define $L(S)$ to be the smallest subspace of V that contains S.

Result 1: W_1 and W_2 are two subspaces of vector space V. $W_1 \cap W_2$ is also a subspace.

Proof :- Take x_1,$x_2 \in W_1 \cap W_2$.
$\Rightarrow x_1 \in W_1$ and $x_1 \in W_2$, $x_2 \in W_1$ and $x_2 \in W_2$
$\Rightarrow x_1 + x_2 \in W_1$ (as $x_1, x_2 \in W_1$ and W_1 is subspace) and $x_1 + x_2 \in W_2$(as $x_1, x_2 \in W_2$ and W_2 is a subspace).
Take $x_1 \in W_1 \cap W_2 \Rightarrow x_1 \in W_1$ and $x_1 \in W_2$
$\Rightarrow \alpha x_1 \in W_1$ and $\alpha x_1 \in W_2$ (as W_1 and W_2 are subspaces)
$\Rightarrow \alpha x_1 \in W_1 \cap W_2$.
Other eight properties can be verified similarly.
$\Rightarrow x_1 + x_2 \in W_1$ and $x_1 + x_2 \in W_2 \Rightarrow x_1 + x_2 \in W_1 \cap W_2$.

Result 2:

W is a subspace of vector space V.S is a non empty subset of $W \Rightarrow L(S) \subset W$.

Proof :-$L(S)$ is the smallest subspace that contains S.$W \cap L(S)$ is a subspace containing S (by using Result 1). $W \cap L(S) \subset L(S)$ this implies we are having a subspace subset of $L(S)$.But $L(S)$ is the smallest subspace containing S. So the only possibility is $W \cap L(S) = L(S)$.
$W \cap L(S) \subset W \Rightarrow L(S) \subset W$

Result 3:

W_1 and W_2 are two subspaces of vector space V.$W_1 \subset W_2$ or $W_2 \subset W_1$ if and only if $W_1 \cup W_2$ is a subspace.

Proof :-$W_1 \subset W_2 \Rightarrow W_1 \cup W_2 = W_2$ Now as W_2 is a subspace this implies $W_1 \cup W_2$ is a subspace.
$W_2 \subset W_1 \Rightarrow W_1 \cup W_2 = W_1$ Now as W_1 is a subspace this implies $W_1 \cup W_2$ is a subspace.
Let us try to prove the converse.
We proceed by contradiction method.$W_1 \cup W_2$ is a subspace. Let us assume that W_1 is not a subset of W_2 and W_2 is not a subset of W_1

$\Rightarrow \exists x \in W_1$ and x is not $\in W_1$ and $\exists y \in W_2$ and y is not $\in W_1$.

$x, y \in W_1 \cup W_2 \Rightarrow x + y \in W_1 \cup W_2$ as $W_1 \cup W_2$ is a subspace.
$\Rightarrow x + y \in W_1$ or $x + y \in W_2$. Let us take the case when $x + y \in W_1$.As W_1 is a subspace $\Rightarrow x + y - x \in W_1$ or $y \in W_1$ which is a contradiction.
When $x + y \in W_2$.As W_2 is a subspace $\Rightarrow x + y - y \in W_2$ or $x \in W_2$ which is a contradiction.
Hence $W_1 \cup W_2$ is a subspace $\Rightarrow W_1 \subset W_2$ or $W_2 \subset W_1$.

Result 4:

Let S be a non-empty set of vector space V and $L(S)$ be the smallest subspace that contains S. W be the set of all possible linear combinations of the elements of S. Then $L(S) = W$.

Proof :-Any element of W can be written as $\alpha x + \beta y$ where $\alpha, \beta \in R$.
Let $x, y \in S$.As $L(S)$ is a subspace $\alpha x \in L(S)$ and $\beta y \in L(S) \Rightarrow \alpha x + \beta y \in L(S)$ $\forall \alpha, \beta \in R$. $\Rightarrow W \subset L(S)$.
Let w_1 and w_2 be two elements of W.Then we can write $w_1 = \alpha_1 x + \beta_1 y$ and $w_2 = \alpha_2 x + \beta_2 y$ where $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$.
$w_1 + w_2 = (\alpha_1 x + \beta_1 y) + (\alpha_2 x + \beta_2 y) \Rightarrow w_1 + w_2 = (\alpha_1 + \alpha_2)x + (\beta_1 + \beta_2)y$
$\Rightarrow w_1 + w_2 = \alpha x + \beta y$ where $\alpha = \alpha_1 + \alpha_2$ and $\beta = \beta_1 + \beta_2$.
$\alpha \in R$ and $\beta \in R$ as $\alpha_1, \alpha_2, \beta_1, \beta_2 \in R$
$\Rightarrow w_1 + w_2 \in W$.
For $w_1 \in W$ and $\alpha' \in R \Rightarrow \alpha' w_1 = \alpha' \alpha_1 x + \alpha' \beta_1 y = \alpha x + \beta y$
So $\alpha' w_1 \in W$. Similarly we can prove other eight properties to show that W is a vector space. $\Rightarrow L(S) \subset W$ as $L(S)$ is the smallest subspace containing S.
$W \subset L(S)$ and $L(S) \subset W \Rightarrow L(S) = W$.

Result 5:

Let V be a vector space $S \subset V$. $s_i, t \in S$, $t = s_j$ (for some $i = j$) $S' = S \setminus \{t\}$. $L(S)$ and $L(S')$ be the smallest subspace containing S and S' respectively. If $\alpha_1 s_1 + \alpha_2 s_2 + ... + \alpha_n s_n = 0$ (all s_i are different) $\Rightarrow \alpha_i = 0$ $\forall i$, then $L(S') \neq L(S)$.

Proof :- $S' \subset S$ and $L(S)$ is a subspace of $V \Rightarrow L(S') \subset L(S)$ (By Result 2).Now we need to show that $L(S') \neq L(S)$. We proceed by contradiction. So let $L(S') = L(S)$. $t \in S \Rightarrow t \in L(S)$(By definition) $\Rightarrow t \in L(S')$ (as $L(S') = L(S)$)
t does not belong to S' but $t \in L(S')$. So we can write t as linear combination of element of S' (by Result 4)
$\Rightarrow t = \alpha_1 s_1 + \alpha_2 s_2 + ... + \alpha_n s_n$ ($t = s_j$ (for some $i = j$))
$\Rightarrow \alpha_1 s_1 + \alpha_2 s_2 + ... + \alpha_n s_n - 1 \cdot t = 0$ ($t = s_j$)
$\Rightarrow \alpha_i \neq 0$(for some s_i) which is a contradiction.

Definition 3:

Let V be a vector space and $S \subset V$. S is said to be linearly independent if $\alpha_1 s_1 + \alpha_2 s_2 + ... + \alpha_n s_n = 0 \Rightarrow \forall i \ \alpha_i = 0$.

Definition 4:

Let V be a vector space and $S \subset V$. S is said to be linearly dependent if it is not linearly independent i.e. $\alpha_1 s_1 + \alpha_2 s_2 + ... + \alpha_n s_n = 0 \Rightarrow \exists i$ such that $\alpha_i \neq 0$.

Result 6 :

$\forall t \in S$ and $L(S \setminus \{t\}) \neq L(S) \Rightarrow S$ is a linearly independent set.

Proof : We will try to prove the converse of the Result. i.e. If S linearly dependent, then $\exists t \in S$ such that $L(S') = L(S)$.
We know that $L(S') \subset L(S)$. We take $t \in S \Rightarrow t \in L(S)$. We can write $t = \alpha_1 s_1 + \alpha_2 s_2 + ... + \alpha_n s_n$ (as S is dependent) where $\forall i\ s_i \in S' \Rightarrow t \in L(S')$ or $L(S) \subset L(S')$.
This completes the proof of converse.

Result 7:

From Result 6 and 7 we have S is linearly independent $\Leftrightarrow \forall t \in S\ L(S \setminus \{t\}) \neq L(S)$.

Definition 5:

Let S be a non empty subset of vector space V is said to be the basis of V if any of the below statement is true.
(a)S is the maximal linearly independent set.
(b)S is the minimal spanning set.
(S is said to be spanning set if $L(S) = V$)
(c)S is linearly independent and spanning set.
(d) $\forall v \in V\ v$ is unique linear combination of elements of S.
All these four statements are equivalent.
Let us try to prove that (a) and (b) are equivalent.

Proof($(a) \Leftrightarrow (b)$) :-Assume S is maximal linearly independent set, Let $u \in V \setminus S$ and $S' = S \cup \{u\} \Rightarrow S'$ is not linearly independent i.e. linearly dependent.
S' is linearly dependent $\Rightarrow \alpha v + \alpha_1 s_1 + ... + \alpha_n s_n = 0\ \exists \alpha, \alpha_i \neq 0$. $\alpha \neq 0$ because if $\alpha = 0$ then it will contradict our assumption that S is linearly independent.
$v = -(\frac{\alpha_1}{\alpha} s_1 + ... + \frac{\alpha_n}{\alpha} s_n) \Rightarrow v \in L(S)$
$V \subset L(S) \Rightarrow L(S) = V$ or S is the spanning set.
S is linearly dependent $\Rightarrow \exists t \in S$ such that $L(S') = L(S)$ (Result 7). This implies that $L(S) = V$ is a minimal spanning set.
For converse spanning set implies maximal and minimal implies linear independence.

Chapter 3

Basics

3.1 Inverse of a function

Definition 1:

$f : X \to Y$, f is called injective or one-to-one if $\forall x_1, x_2 \in X \Rightarrow f(x_1) \neq f(x_2)$. In terms of converse if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Definition 2:

f is said to be surjective or onto if $\forall y \in Y$ $\exists$ $x \in X$ such that $y = f(x)$.

Definition 3:

f is said to be bijective if f is injective and surjective.

Definition 4:

f^{-1} exists means $\exists$ $g : Y \to X$ such that $g \cdot f = x$ and $f \cdot g = y$, then we call g as the inverse of f or simply $g = f^{-1}$.

Result 1:

f is said to be bijective iff f^{-1} exists.

Proof :- From definition of function it is easy to show that if f is bijective then f^{-1} exists. Let us try to prove the converse by Definition 4.
f^{-1} exists means there exists $g : Y \to X$ such that $g \cdot f = x$ and $f \cdot g = y$.
Take $g \cdot f = x$. If $f(x_1) = f(x_2) \Rightarrow g(f(x_1)) = g(f(x_2)) \Rightarrow x_1 = x_2$
Also we can try the converse which is as follows
If $x_1 \neq x_2 \Rightarrow g(f(x_1)) \neq g(f(x_2)) \Rightarrow f(x_1) \neq f(x_2)$ (as g is a function)
This confirms that f is injective.
By the definition of f $\forall x x \mapsto f(x)$. $x = g(f(x)) = g(y)$ $\forall y \in Y$ there exists $x \in X$ in away that $x \mapsto f(x)$ such that $x = g(y)$.
This implies g is surjective.

Similarly by taking $f \cdot g = y$ we can prove that f is surjective and g is injective. Hence it confirms that f is bijective.

Definition 5:

$f : X \to Y$ Fibre of (a point) $y \in Y$ is denoted by $f^{-1}(\{y\})$ and is defined as $f^{-1}(\{y\}) = \{x \in X/f(x) = y\}$.

Result 2:

(a)A function f is said to be injective if all the Fibre sets are either empty or singleton. or If any horizontal line intersect graph then it must intersect at only one point.
(b)A function f is said to be surjective if all the Fibre sets are non-empty. or Any horizontal line will intersect the graph of f at some point.
(c)A function f is said to be bijective if all the Fibre sets are singleton. or Any horizontal line intersect f at only one point.

3.2 Geometrical interpretation of solving three variable two homogeneous equation

Consider a system of two equations $a_1x+b_1y+c_1z = 0$ (P_1) and $a_2x+b_2y+c_2z = 0$ (P_2) with $c_2 \neq 0$ and $P_1 \neq \alpha P_2$ for some $\alpha \in R$. We are interested in finding the solution of this system geometrically. By method of gauss elimination we multiply $\frac{-c_1}{c_2}$ with P_2 and add it with P_1 to get the equation of another plane P_1'. $P_1' = d_1x + d_2y = 0$.(With either d_1 or d_2 non zero).Firstly we have to show that the intersection of P_1 and P_2 is same as the intersection of $P_1' = P_1 + \lambda P_2$ (for some $\lambda \in R$) and P_2. Algebraically it is easy to show that. Let us try geometrically.
P_1 and P_2 both pass through origin. Let L be the intersection of P_1 and P_2 which is a line(as possibility of these two planes to be identical we have eliminated by $P_1 \neq \alpha P_2$ for some $\alpha \in R$).
$\Rightarrow$ L passes through origin. Let L' be the intersection of P_1' and P_2. P_1' is a plane that contains the Z-axis as coefficient of z in its equation is 0. We can have a plane containing the Z-axis and L (as both pass through origin i.e. intersect).P_1' contains Z-axis. Now if we take $L' = L$ then the required plane will be P_1' itself.
After we get the equation of P_1' we write y in terms of x $(d_2 \neq 0)$. and back substitute this value of y and x in equation of P_2 to get the value of z. Lets see what happens geometrically.
Equation of P_1' will look like a straight line with slope $\frac{-d_1}{d_2}$ in XY- plane. This straight line is the projection of P_1' into XY- plane. Whenever we see projection of a plane into XY -plane we are supposed to get the whole XY plane. But by doing Gaussian elimination we get a line as the projection. Now this line is the protection of the line which is contained in P_1'. In Gaussian elimination when we back substitute the value of y and x we get the corresponding z. x and y

satisfy the equation of P_1'. When we do back substitution we get the line L as we back substitute in the equation of P_2 which contains L. Geometrically we re project the projected line to the original line L.

3.3 Lines

Definition 1:

Let V be a vector space and $0 \neq d \in V$. A line passing through $p \in V$ and having $dirctiond$, is denoted by $l(p; d)$ and defined as
$l(p; d) = \{v \in V/$ there exists $t \in R$, $v = p + td\} = p + Rd$
d is called the direction vector.as the Line is the unlimited extension of the line segment joining 0 and d.

Result 1:

$l(p; d) = l(q; d)$ iff $(q - p)$ is a multiple of d.
Proof :- Let $l(p; d) = l(q; d)$. Take $x \in l(p; d) \Rightarrow x \in l(q; d)$ as both the lines are same. Then there exists $s, t \in R$ such that $x = p+sd = q+td$.So $q-p = (s-t)d$, that is,$q - p$ is a multiple of d.
Conversely let $q - p$ is a multiple of d. It means $q - p = \alpha d$ for some $\alpha \in R$. Let $v \in l(p; d)$, then $v = p + td = (q - \alpha d) + td = q + (t - \alpha)d$ for some $t \in R$. Therefore $v \in l(q; d)$ and $l(p; d) \subseteq l(q; d)$. Similarly we can prove $l(q; d) \subseteq l(p; d)$ and hence $l(p; d) = l(q; d)$.

Result 2:

$l(p; d) = l(p; \alpha d)$ for any $\alpha \in R \setminus \{0\}$.

Proof :-$l(p; d) \subset l(p; \alpha d)$ for $\alpha = 1$. Take a point $x \in l(p; \alpha d)$, then there exists $t \in R$ such that $x = p + t\alpha d$. We choose $s = \alpha t$ for $t \in R$ such that $x = p + \frac{s}{\alpha} \cdot \alpha d$ or $x = p + sd$. $\alpha \neq 0$ means for a particular t we can choose the corresponding s only(Our choice is a bijection map). It means for every existing $t \in R$ we can find corresponding existing $s \in R$ such that $x = p + sd$. $\Rightarrow x \in l(p; d)$ or $l(p; \alpha d) \subset l(p; d)$. This completes the proof.

Result 3:

$l(p; d) = l(q; d)$ for any $q \in l(p; d)$.

Proof :-$l(p; d) \subset l(q; d)$ by taking $q = p \in l(p; d)$. For $q \in l(p; d)$ we can write there exists $t \in R$ such that $q = p+td$...(i). Take $r \in l(q; d)$. We can write that there exists $s \in R$ such that $r = q + sd$...(ii).
$\Rightarrow$ There exists $t, s \in R$ such that $r = p + (t + s)d$(by adding (i) and (ii)). We choose $u = t + s$ for $t, s \in R$. For any particular t and s we can choose the corresponding u only(Our choice is a bijection map).So for every existing t, s we can find corresponding $u \in R$ such that $r = p + ud$. $\Rightarrow r \in l(p; d)$ or $l(q; d) \subset l(p; d)$. This completes the proof.

Result 4:

Any two distinct point determine a unique line.

Proof :-Let $l(p;d)$ be a line that passes through p and q. Then there exists $t \in R$ such that $q = p + td$..(i). Let $l(r,d')$ be another line that contains p and q. Then there exists $t_1, t_2 \in R$ such that $p = r + t_1 d'$..(ii) and $q = r + t_2 d'$ ..(iii). (iii)-(ii) yields $q = p + (t_2 - t_1)d'$..(iv). As p and q are different we can not have $t_2 - t_1 = 0$ So $q \in l(p;d')$. Now (iv)-(i) yields $(t_2 - t_1)d' = td$ By dividing both sides by $t_2 - t_1$ we have $d' = \alpha d$ as multiple of d (Note $t \neq 0$ as p and q are different $\Rightarrow \alpha \neq 0)$. So $p, q \in p(l; \alpha d) \Rightarrow p, q \in l(p;d)$ (by result 3) It is the same line.

Definition 2:

We say two lines $l(p;d_1) = l(p,d_2)$ are parallel if $d_1 = \alpha d_2$ for some $\alpha \in R$. $(\alpha, d_1, d_2 \neq 0)$

Result 5:

Given l and q not belonging to l there exists a unique line $l(q;d)$ such that $l(q;d)$ is parallel to l.

Proof :- Let d be the direction of l. Then $l(q;d)$ is passing through q and parallel to l. Let there exists another line $l(q;d_1)$ such that $l(q;d_1)$ is parallel to l. Then $d_1 = \alpha d$ for some $\alpha \in R \setminus \{0\}$. From Result 2 it follows that $l(q;d_1) = l(q;d)$.